All 4-Edge-Connected HHD-Free Graphs are Z3 -Connected

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All 4-Edge-Connected HHD-Free Graphs are $\mathbb{Z}_3$-Connected

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Abstract

An undirected graph $G = (V, E)$ is called $\mathbb{Z}_3$-connected if for all $b : V \to \mathbb{Z}_3$ with $\sum_{v \in V} b(v) = 0$, an orientation $D = (V, A)$ of G has a $\mathbb{Z}_3$-valued nowhere-zero flow $f : A \to \mathbb{Z}_3 - \{0\}$ such that $\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = b(v)$ for all $v \in V$. We show that all 4-edge-connected HHD-free graphs are $\mathbb{Z}_3$-connected. This extends the result due to Lai (2000), which proves the $\mathbb{Z}_3$-connectivity for 4-edge-connected chordal graphs.
Keywords: edge-connectivity, HHD-free graph, nowhere-zero flow, $\mathbb{Z}_3$-connectivity

1 Introduction

Let $D = (V, A)$ be a digraph with vertex set V and arc set A. For a vertex $v \in V$, $\delta^+(v)$ (resp., $\delta^-(v)$) denotes the set of arcs leaving (resp., entering) v. Let $b : V \to \mathcal{A}$, where $\mathcal{A}$ denotes an abelian group with identity 0. When $\sum_{v \in V} b(v) = 0$, b is called a *zero-sum function*. For a zero-sum function b, a *nowhere-zero* $(\mathcal{A}, b)$*-flow* is defined as a function $f : A \to \mathcal{A} - \{0\}$ such that $\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = b(v)$ for all $v \in V$. Notice that the existence of nowhere-zero $(\mathcal{A}, b)$-flows only depends on b and the underlying undirected graph $G = (V, E)$ of D because reversing the direction of an arc $e \in A$ corresponds to reversing the sign of $f(e)$.

An undirected graph G is called $\mathcal{A}$*-connected* when its orientation (i.e., a digraph obtained by directing edges) has a nowhere-zero $(\mathcal{A}, b)$-flow for each zero-sum function $b : V \to \mathcal{A}$. Let $\mathbb{Z}_3$ denote the cyclic group of order 3. This paper is concerned with the following conjecture due to Jaeger et al. [2].

Conjecture 1 ([2]). *Every 5-edge-connected undirected graph is $\mathbb{Z}_3$-connected.*

This conjecture is closely related to the 3-flow conjecture due to Tutte [3], which is a long-standing open problem in graph theory. A nowhere-zero 3-flow of a digraph $D = (V, A)$ is defined as $f : A \to \{\pm 1, \pm 2\}$ such that $\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = 0$ for all $v \in V$. The 3-flow conjecture can be stated as follows.

Conjecture 2 ([3]). *Every orientation of 4-edge-connected undirected graphs has a nowhere-zero 3-flow.*

It is known that a nowhere-zero 3-flow exists if and only if a nowhere-zero $(\mathcal{A}, b)$-flow exists for $|\mathcal{A}| = 3$ and $b(v) = 0$, $v \in V$. Kochol [4] has shown that if every 5-edge-connected undirected graph has a nowhere-zero 3-flow, then every 4-edge-connected undirected graph does. This means that Conjecture 1 implies the 3-flow conjecture. Refer to [5, 6] for more information on the 3-flow conjecture.

Motivated by Conjecture 1, there are several works investigating which graphs are $\mathbb{Z}_3$-connected [7, 8]. For example, Lai [9] characterized 3-edge-connected chordal graphs which are $\mathbb{Z}_3$-connected. As a corollary of his characterization, it can be shown that all 4-edge-connected chordal graphs are $\mathbb{Z}_3$-connected. A simple proof for this fact will be presented in Section 2. We note that the Lai's result was extended to triangularly connected graphs by Fan et al. [1].

Our main contribution in this paper is to show the $\mathbb{Z}_3$-connectivity of HHD-free graphs.

Theorem 1. *Every 4-edge-connected HHD-free graph is $\mathbb{Z}_3$-connected.*

HHD-free graphs, which will be defined in Section 2, form a super-class of chordal graphs. Thus Theorem 1 extends the $\mathbb{Z}_3$-connectivity of 4-edge-connected chordal graphs.

Let H be a subgraph of G, and G/H denote the graph obtained from G by contracting the edges in H and removing the generated loops. For proving $\mathcal{A}$-connectivity of graphs in a family closed under contraction, it suffices to show that each graph in the family has $\mathcal{A}$-connected subgraphs by the following theorem.

Theorem 2. *Let H be an $\mathcal{A}$-connected subgraph of an undirected graph G. If G/H is $\mathcal{A}$-connected, then G is also $\mathcal{A}$-connected.*

Actually most of the previous works on $\mathcal{A}$-connectivity are based on this observation. Its proof can be found in many papers, for example [9]. This paper also takes the same approach with them. We prove Theorem 1 by showing that every non-trivial 4-edge-connected HHD-free graph contains a non-trivial $\mathbb{Z}_3$-connected subgraph.

The rest of this paper is organized as follows. In Section 2, we introduce notations and basic facts on graphs. We also present a simple proof for $\mathbb{Z}_3$-connectivity of 4-edge-connected chordal graphs. In Section 3, we prove Theorem 1.

2 Preliminaries on Graphs and Nowhere-zero Flows

For an undirected graph G, $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. A graph G is called *trivial* if $|V(G)| = 1$, and *non-trivial* otherwise. A trivial graph is obviously $\mathbb{Z}_3$-connected. When $E(G) = \{e\}$, $V(G)$ may be represented by $V(e)$. For $u, v \in V(G)$, uv denotes the undirected edge in $E(G)$ joining vertices u and v. In contrast, an arc with u as the tail and with v as the head is denoted by $\overrightarrow{uv}$. For $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U. For a vertex v, $N(v)$ denotes the set of neighbors of v.

We let C_k stand for the cycle on k vertices, and P_k stand for the path obtained by removing one edge from C_k. A path P_k with $V(P_k) = \{v_1, v_2, \ldots, v_k\}$ and $E(P_k) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$ is represented by v_1 - v_2 - $\cdots$ - v_k. We let W_k denote a graph obtained by adding one vertex to C_k with edges joining the vertex and each vertex in $V(C_k)$.

Theorem 3. *C_2 and W_4 are $\mathbb{Z}_3$-connected.*

The proof of Theorem 3 appears in several papers. Refer to [9] for example.

A graph is called *chordal* if it contains no induced subgraph isomorphic to C_k with $k \geq 4$. A vertex v is called *simplicial* if $G[N(v)]$ is complete. In other words, it is defined as a vertex which is not a midpoint of any induced P_3. Chordal graphs can be characterized by the existence of simplicial vertices as follows.

Theorem 4 ([10])**.** *Every chordal graph has a simplicial vertex.*

We can observe that this characterization presents a simple proof of the theorem appeared in [9].

Theorem 5. *Every 4-edge-connected chordal graph is $\mathbb{Z}_3$-connected.*

Proof. First of all, let us mention that the family of 4-edge-connected chordal graphs is closed under contracting edges. It is immediate from the definition that the contraction does not decrease the edge-connectivity. If a graph obtained by contracting an edge is not chordal, then it contains an induced C_k with some $k \geq 4$. In the original graph, this cycle is an induced C_k, or a P_k connecting the end vertices of the contracted edge. In the latter case, the original graph contains an induced C_{k+1}. Hence the original graph is not chordal if the graph after the contraction is not chordal.

Let G be a counter-example of Theorem 5 (i.e., G is a 4-edge-connected chordal graph that is not $\mathbb{Z}_3$-connected) minimizing $|V(G)| + |E(G)|$. G is non-trivial because the trivial graphs are obviously $\mathbb{Z}_3$-connected. We show that G contains a non-trivial connected subgraph which is $\mathbb{Z}_3$-connected. If such a subgraph exists, contracting it gives a smaller graph, which is $\mathbb{Z}_3$-connected by the definition of G. Then by Theorem 2, G is $\mathbb{Z}_3$-connected.

Parallel edges form a $\mathbb{Z}_3$-connected subgraph C_2 of G. Hence let us suppose that G is simple. Let v^* be a simplicial vertex in G, whose existence is guaranteed by Theorem 4. Since G is simple and 4-edge-connected, $|N(v^*)| \geq 4$. Moreover, because the subgraph induced by $N(v^*)$ is complete by the definition of v^*, it contains a C_4. Hence the subgraph induced by $N(v^*) \cup \{v^*\}$ contains a W_4, which is $\mathbb{Z}_3$-connected. □

The *house* stands for the graph consisting of five vertices and six edges where a C_4 and a C_3 share one edge (See Fig. 1). The *domino* stands for the graph consisting of six vertices and seven edges where two C_4's share one edge (See Fig. 1). A graph is called *HHD-free* if any C_k with $k \geq 5$ in the graph has at least two chords. This is equivalent to containing no

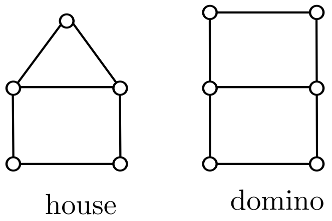


Figure 1: house and domino

induced subgraph isomorphic to house, domino, and C_k with $k \geq 5$. Similarly for chordal graphs, HHD-free graphs can be characterized by the existence of special vertices. A vertex v is called *semi-simplicial* if v is not a midpoint of any induced P_4.

Theorem 6 ([11])**.** *Every HHD-free graph has a semi-simplicial vertex.*

In the next section, we see that this characterization is useful for proving the Z_3-connectivity of 4-edge-connected HHD-free graphs as Theorem 4 is useful for proving Theorem 5.

Let us introduce several tools for proving $\mathbb{Z}_3$-connectivity. Let xv and yv be two undirected edges incident to a vertex v. *Splitting off* $\{xv, yv\}$ denotes the operation of replacing xv and yv by a new edge xy.

Theorem 7. *Let e_1 and e_2 be two edges incident to the same vertex in an undirected graph G, and G' be the graph obtained from G by splitting off $\{e_1, e_2\}$. If G' is $\mathbb{Z}_3$-connected, then G is $\mathbb{Z}_3$-connected.*

Theorem 8. *Let v be a vertex of even degree in an undirected graph G, and G' be the graph obtained from G by repeatedly splitting off the edges incident to v until no edge is incident to v and removing v. If G' is $\mathbb{Z}_3$-connected, then an orientation of G has a nowhere-zero $(\mathbb{Z}_3, b)$-flow for any zero-sum function $b : V(G) \to \mathbb{Z}_3$ such that $b(v) = 0$.*

Refer to [9] for proofs of Theorems 7 and 8.

Theorem 9. *Let v be a vertex of degree 3 in an undirected graph G, and G' be the graph obtained from G by splitting off one pair of edges incident to v and removing v (with the last edge incident to v). If G' is $\mathbb{Z}_3$-connected, then an orientation of G has a nowhere-zero $(\mathbb{Z}_3, b)$-flow for any zero-sum function $b : V(G) \to \mathbb{Z}_3$ such that $b(v) \in \{1, 2\}$.*

Proof. Let x_1v, x_2v, and x_3v be the edges incident to v in G, and G' is obtained by splitting off $\{x_1v, x_2v\}$, and deleting v and x_3v. Define $b' : V(G') \to \mathbb{Z}_3$ by $b'(u) = b(u)$ for $u \in V(G') \setminus \{x_3\}$, and $b'(x_3) = b(x_3) + b(v)$. Since $\sum_{u \in V(G')} b'(u) = \sum_{u \in V(G)} b(u) = 0$, b' is a zero-sum function on $V(G')$.

Let D' be an orientation of G' such that the edge x_1x_2 generated by splitting off $\{x_1v, x_2v\}$ is oriented from x_1 to x_2. Since G' is $\mathbb{Z}_3$-connected, D' has a nowhere-zero $(\mathbb{Z}_3, b')$-flow f'.

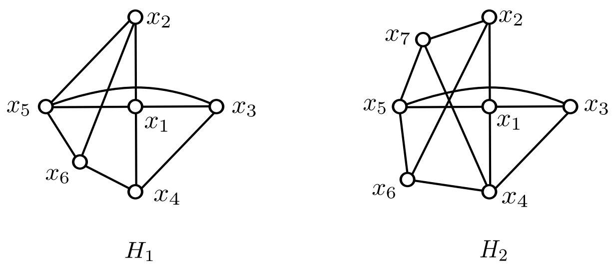


Figure 2: Z_3-connected graphs used for proving Theorem 1

Let D be an orientation of G such that the edges not incident to v have the same orientation with D', x_1v is oriented from x_1 to v, x_2v is oriented from v to x_2, and x_3v is oriented from v to x_3. Then D has a nowhere-zero $(\mathbb{Z}_3, b)$-flow f defined as follows;

$$f(e) = \begin{cases} f'(e) & \text{for } e \text{ not incident to } v, \\ f'(\overrightarrow{x_1x_2}) & \text{for } e \in \{\overrightarrow{x_1v}, \overrightarrow{vx_2}\}, \\ b(v) & \text{for } e = \overrightarrow{vx_3}. \end{cases}$$

□

For proving Theorem 1, we need to prove the $\mathbb{Z}_3$-connectivity of two specific graphs. We define graphs H_1 and H_2 as in Fig. 2.

Lemma 1. *H_1 is $\mathbb{Z}_3$-connected.*

Proof. In the proof, we see that an orientation of H_1 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow for every zero-sum function $b : V(H_1) \to \mathbb{Z}_3$. Notice that the adjacency among the vertices in H_1 does no change even when we simultaneously let x_1 exchange its name with x_5, x_2 with x_3, and x_4 with x_6.

First of all, consider the case where $b(x_1) = 0$. Split off $\{x_1x_3, x_1x_4\}$ and $\{x_1x_2, x_1x_5\}$, and delete x_1. The graph obtained by this operation is $\mathbb{Z}_3$-connected by Theorems 2 and 3 because, by repeatedly contracting generated C_2's, we obtain a trivial graph. Hence by Theorem 8, an orientation of H_1 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow in this case. Because of the symmetry between x_1 and x_5, we can similarly prove the existence of nowhere-zero $(\mathbb{Z}_3, b)$-flows when $b(x_5) = 0$.

Let us consider the case when $b(x_2) \in \{1, 2\}$ in the next. Split $\{x_1x_2, x_2x_5\}$ off, and delete x_2. The graph obtained by this operation is $\mathbb{Z}_3$-connected again because, by repeatedly contracting generated C_2's, we obtain a trivial graph. Hence by Theorem 9, an orientation of H_1 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow in this case. Because of the symmetry between x_2 and x_3, we can similarly prove the existence of nowhere-zero $(\mathbb{Z}_3, b)$-flows when $b(x_3) \in \{1, 2\}$.

Let $b(x_4) \in \{1, 2\}$. In this case, split off $\{x_1x_4, x_3x_4\}$ and delete x_4. The graph obtained by this operation is $\mathbb{Z}_3$-connected because, by repeatedly contracting generated C_2's, we obtain a trivial graph. Hence by Theorem 9, an orientation of H_1 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow in this case. Because of the symmetry between x_4 and x_6, we can similarly prove the existence of nowhere-zero $(\mathbb{Z}_3, b)$-flows when $b(x_6) \in \{1, 2\}$.

The remaining case is when $b(x_1), b(x_5) \in \{1, 2\}$ and $b(x_2) = b(x_3) = b(x_4) = b(x_6) = 0$. By the fact that b is zero-sum and the symmetry between x_1 and x_5, it suffices to consider when $b(x_1) = 1$ and $b(x_5) = 2$. Let D be the digraph obtained by orienting each edge in H_1 from the vertex of smaller index to the other. For this function b, D has a nowhere-zero $(\mathbb{Z}_3, b)$-flow f defined by

$$f(e) = \begin{cases} 1 & \text{if } e \in \{\overrightarrow{x_1x_3}, \overrightarrow{x_2x_5}, \overrightarrow{x_2x_6}, \overrightarrow{x_4x_6}, \overrightarrow{x_5x_6}\}, \\ 2 & \text{otherwise.} \end{cases}$$

□

Lemma 2. *H_2 is $\mathbb{Z}_3$-connected.*

Proof. We show that an orientation of H_2 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow for every zero-sum function $b : V(H_2) \to \mathbb{Z}_3$.

For the case when $b(x_1) = 0$, split off $\{x_1x_3, x_1x_4\}$ and $\{x_1x_2, x_1x_5\}$, delete x_1, and contract the generated C_2. Then we obtain a W_4, which is $\mathbb{Z}_3$-connected. Hence by Theorem 8, an orientation of H_2 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow in this case.

In the next, let us consider the case when $b(x_2) \in \{1, 2\}$. Split $\{x_1x_2, x_2x_6\}$ off and delete x_2. Then the obtained graph contains a W_4 (induced by $\{x_1, x_3, x_4, x_5, x_6\}$). By contracting this W_4, we obtain a C_2, which is $\mathbb{Z}_3$-connected. This means that an orientation of H_2 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow when $b(x_2) \in \{1, 2\}$ by Theorem 9.

If $b(x_3) \in \{1, 2\}$, split $\{x_1x_3, x_3x_4\}$ off, delete x_3, and contract the generated C_2. Then we obtain a W_4, which is $\mathbb{Z}_3$-connected. Thus by Theorem 9, an orientation of H_2 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow when $b(x_3) \in \{1, 2\}$.

If $b(x_6) \in \{1, 2\}$, split $\{x_2x_6, x_5x_6\}$ off and delete x_6. Then the obtained graph is isomorphic to H_1. By Lemma 1, this is $\mathbb{Z}_3$-connected. Thus Theorem 9 implies that an orientation of H_2 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow when $b(x_6) \in \{1, 2\}$.

Next, consider the case when $b(x_4) = b(x_5) = 0$. In this case, split off $\{x_1x_4, x_4x_6\}$, $\{x_3x_4, x_4x_7\}$, $\{x_3x_5, x_5x_7\}$ and $\{x_1x_5, x_5x_6\}$, and delete x_4 and x_5. Then contract the generated C_2's repeatedly. At last, we obtain a trivial graph, which is $\mathbb{Z}_3$-connected. Hence Theorem 8 implies that an orientation of H_2 has a nowhere-zero $(\mathbb{Z}_3, b)$-flow when $b(x_4) = b(x_5) = 0$.

The remaining case is when $b(x_1) \in \{1, 2\}$, $b(x_2) = b(x_3) = b(x_6) = b(x_7) = 0$, and at most one of $b(x_4)$ and $b(x_5)$ is 0. Notice that x_4 and x_5 have the same neighbors. By this symmetry, it suffices to consider $(b(x_1), b(x_4), b(x_5)) \in \{(1, 0, 2), (1, 1, 1), (2, 0, 1), (2, 2, 2)\}$.

Let D be the digraph obtained by orienting each edge in H_2 from the vertex of smaller index to the other. For $(b(x_1), b(x_2), \ldots, b(x_7)) = (2, 0, 0, 0, 1, 0, 0)$, D has a nowhere-zero

$(\mathbb{Z}_3, b)$-flow f defined as

$$f(e) = \begin{cases} 1 & \text{if } e \in \{\overrightarrow{x_1x_2}, \overrightarrow{x_1x_3}, \overrightarrow{x_1x_5}\}, \\ 2 & \text{otherwise.} \end{cases}$$

By exchanging the flow values 1 and 2 of f for all arcs, we also obtain a nowhere-zero $(\mathbb{Z}_3, b)$-flow for $(b(x_1), b(x_2), \dots, b(x_7)) = (1, 0, 0, 0, 2, 0, 0)$.

For $(b(x_1), b(x_2), \dots, b(x_7)) = (2, 0, 0, 2, 2, 0, 0)$, D has a nowhere-zero $(\mathbb{Z}_3, b)$-flow f' defined as

$$f'(e) = \begin{cases} 1 & \text{if } e \in \{\overrightarrow{x_1x_2}, \overrightarrow{x_1x_4}, \overrightarrow{x_1x_5}, \overrightarrow{x_3x_5}, \overrightarrow{x_3x_5}\}, \\ 2 & \text{otherwise.} \end{cases}$$

By exchanging the flow values 1 and 2 of f' for all arcs, we also obtain a nowhere-zero $(\mathbb{Z}_3, b)$-flow for $(b(x_1), b(x_2), \dots, b(x_7)) = (1, 0, 0, 1, 1, 0, 0)$. $\square$

3 Proof of Theorem 1

Let us see that the family of 4-edge-connected HHD-free graphs is closed under contracting edges. Suppose that a graph obtained by contracting an edge is not HHD-free. It then contains a subgraph isomorphic to C_k with at most one chord for some $k \geq 5$. In the original graph, this subgraph is a C_k or a P_k connecting the end vertices of the contracted edge, with at most one chord. In the latter case, the original graph contains a C_{k+1}. Hence the original graph is not HHD-free if the graph after the contraction is not HHD-free.

Let G be a counter-example of Theorem 1 (i.e., G is a 4-edge-connected HHD-free graph that is not $\mathbb{Z}_3$-connected) minimizing $|V(G)| + |E(G)|$. As seen in the proof of Theorem 5, G is non-trivial and simple. Moreover, G is not chordal by Theorem 5. In what follows, we show that G contains a connected subgraph which is $\mathbb{Z}_3$-connected. Contracting the subgraph gives a smaller graph, which is $\mathbb{Z}_3$-connected by the definition of G. Then by Theorem 2, G is $\mathbb{Z}_3$-connected, which proves Theorem 1

Let v^* be a semi-simplicial vertex in G, whose existence is guaranteed by Theorem 6. Let us consider the case where $N(v^*) \cup \{v^*\} = V(G)$. Since G is not chordal, it contains an induced subgraph H isomorphic to C_4. The subgraph H does not contain v^* because all vertices in $V(G) - \{v^*\}$ are adjacent to v^*. This means that $G[V(H) \cup \{v^*\}]$ is a W_4, which is $\mathbb{Z}_3$-connected.

In the remainder of this section, let us suppose that $N(v^*) \cup \{v^*\} \neq V(G)$. That is to say, G contains a vertex not in $N(v^*) \cup \{v^*\}$. We let O denote the set of vertices in $V(G) - (N(v^*) \cup \{v^*\})$ that are adjacent to some vertex in $N(v^*)$. By the 4-edge-connectivity of G, there exist at least four edge-disjoint paths between v^* and vertices in O. We call a set $\mathcal{P}$ of such edge-disjoint paths *minimal* if for each $P \in \mathcal{P}$, G does not contain another path P' between v^* and vertices in O such that $V(P') \cap N(v^*) \subset V(P) \cap N(v^*)$ and P' is edge-disjoint from all those in $\mathcal{P} - \{P\}$.

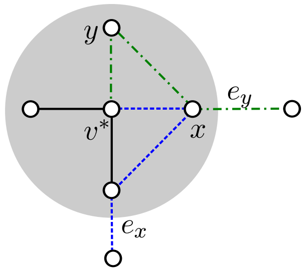


Figure 3: An example for $N(e_x) \neq x$ and $x = N(e_y)$. The dotted lines represent P_x and the chain lines represent P_y.

Choose a minimal set $\mathcal{P} = \{P_x, P_y, P_z, P_w\}$ of four edge-disjoint paths between v^* and vertices in O, where x, y, z and w denote the vertex next to v^* in P_x, P_y, P_z and P_w, respectively. Notice that x, y, z and w are all different vertices because G is simple. For $v \in \{x, y, z, w\}$, let e_v denote the edge in $E(P_v)$ joining a vertex in $N(v^*)$ and one in O while $O(e_v)$ denotes the vertex in $O \cap V(e_v)$ and $N(e_v)$ denotes the vertex in $N(v^*) \cap V(e_v)$. Moreover, let $O(\mathcal{P}) = \{O(e_v) \mid v \in \{x, y, z, w\}\}$ and $N(\mathcal{P}) = \{N(e_v) \mid v \in \{x, y, z, w\}\}$.

When there are several choices of $\mathcal{P}$, we choose one that minimizes $|O(\mathcal{P})|$. If there are two minimal sets $\mathcal{P}' = \{P'_{x'}, P'_{y'}, P'_{z'}, P'_{w'}\}$ and $\mathcal{P}'' = \{P''_{x''}, P''_{y''}, P''_{z''}, P''_{w''}\}$ such that $|O(\mathcal{P}')| = |O(\mathcal{P}'')| = 2$, $O(P'_{x'}) = O(P'_{y'}) = O(P'_{z'}) \neq O(P'_{w'})$ and $O(P''_{x''}) = O(P''_{y''}) \neq O(P''_{z''}) = O(P''_{w''})$, then we give $\mathcal{P}'$ priority over $\mathcal{P}''$.

By the minimality of $\mathcal{P}$, the following observation holds.

Lemma 3. $N(\mathcal{P}) \subseteq \{x, y, z, w\}$.

Proof. Suppose that $N(e_x) \notin \{x, y, z, w\}$ without loss of generality. Let P' denote the path v^* - $N(e_x)$ - $O(e_x)$. Then $V(P') \subset V(P_x)$ and P' is edge-disjoint from P_y, P_z, and P_w. This contradicts the minimality of $\mathcal{P}$. $\square$

We also assume that every $v \in N(\mathcal{P})$ satisfies $N(e_v) = v$. For observing that this assumption preserves the generality, let us consider the case where $x \in N(\mathcal{P})$ and $N(e_x) \neq x$ for example. $x \in N(\mathcal{P})$ means that $x = N(P_v)$ for some $v \in \{y, z, w\}$. (Fig. 3 shows an example where $x = N(P_y)$). Then we can exchange the subpath of P_x from x to $O(P_x)$ for e_v of P_v while preserving the conditions on $\mathcal{P}$. By repeating this, we can satisfy the condition on $\mathcal{P}$.

In the following, we classify cases according to $\mathcal{P}$.

When $|O(\mathcal{P})| = 4$

Let $u \in N(\mathcal{P})$ (i.e., $N(e_u) = u$), and $v \in N(v^*) - \{u\}$. Then v^* is a midpoint of v - v^* - u - $O(e_u)$. Notice that $O(e_u)$ is not adjacent to v^* since $O(e_u) \notin N(v^*)$, and to v since otherwise one

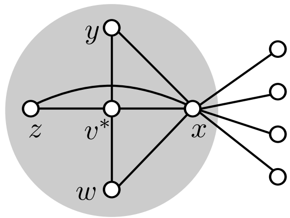


Figure 4: An example for $|O(\mathcal{P})| = 4$ and $N(\mathcal{P}) = \{x\}$

path in $\mathcal{P}$ can be replaced by another path v^* - v - $O(e_u)$, which decreases $|O(\mathcal{P})|$. Hence $E(G)$ must contain an edge uv for forbidding the P_4 from being an induced subgraph.

This fact implies that if $|N(\mathcal{P})| \geq 2$, then $G[\{x, y, z, w\}]$ contains a C_4, and hence $G[\{v^*, x, y, z, w\}]$ contains a W_4. Now let us consider the case where we cannot take such $\mathcal{P}$. Without loss of generality, $N(\mathcal{P}) = \{x\}$ (See Fig. 4), and x is a cut-vertex (i.e., no edge exists between vertices in $N(v^*) - \{x\}$ and those in O). Then $G[N(v^*) \cup \{v^*\}]$ is a 4-edge-connected HHD-free graph which has a semi-simplicial vertex adjacent to all the other vertices. We have already seen that such a graph is $\mathbb{Z}_3$-connected.

When $|O(\mathcal{P})| = 3$

Let us consider the case where $O(e_x) = O(e_y)$ without loss of generality. Call $O(e_x) = O(e_y)$ by o. Since G is simple, $N(e_x) \neq N(e_y)$. Hence $|N(\mathcal{P})| \geq 2$.

Let $u \in \{N(e_x), N(e_y)\}$ and $v \in \{z, w\} \setminus \{N(e_x), N(e_y)\}$. Notice that no edge joins o and v since otherwise, there exists another set $\mathcal{P}'$ of four edge-disjoint paths with $|O(\mathcal{P}')| = 2$. Hence for forbidding the P_4 v - v^* - u - o from being an induced subgraph, there exist an edge uv.

Next, let $u \in N(\mathcal{P})$ and $v \in \{x, y, z, w\} \setminus N(\mathcal{P})$. Since $N(e_u) = u$, there exists a P_4 v - v^* - u - $O(e_u)$. Notice that no edge joins v and $O(e_u)$ since otherwise, it contradicts the minimality of $\mathcal{P}$. Hence for forbidding the P_4 from being an induced subgraph, there exists an edge uv.

If $\{x, y\} \subseteq N(\mathcal{P})$, then apply the first argument for all pairs of $u \in \{x = N(e_x), y = N(e_y)\}$ and $v \in \{z, w\}$. If $|N(\mathcal{P})| = 2$, then apply the second argument for all pairs of $u \in N(\mathcal{P})$ and $v \in \{x, y, z, w\} \setminus N(\mathcal{P})$. If these two cases do not happen, i.e., $|\{x, y\} \cap N(\mathcal{P})| = 1$, $\{z, w\} \subseteq N(\mathcal{P})$ and $|\{N(e_x), N(e_y)\} \cap \{z, w\}| = 1$, then apply the first argument for all pairs of $u \in \{N(e_x), N(e_y)\}$ and $v \in \{z, w\} \setminus \{N(e_x), N(e_y)\}$, and apply the second argument for all pairs of $u \in \{N(e_x), N(e_y)\}$ and $v \in \{x, y\} \setminus \{N(e_x), N(e_y)\}$. In any cases, we obtain a C_4 included in $G[\{x, y, z, w\}]$, and hence a W_4 included in $G[\{v^*, x, y, z, w\}]$ (See Fig. 5).

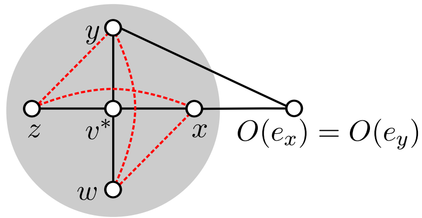
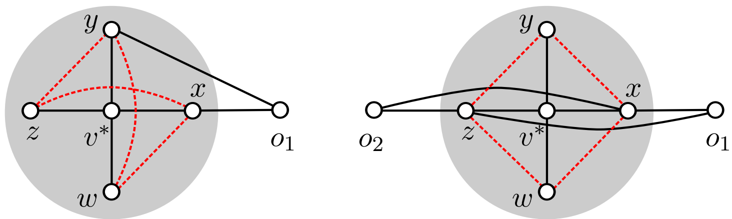


Figure 5: An example for $|O(\mathcal{P})| = 3$ and $O(e_x) = O(e_y)$



(a) (b)

Figure 6: Examples for $|O(\mathcal{P})| = 2$ and $O(e_x) = O(e_y) \neq O(e_z) = O(e_w)$

When $|O(\mathcal{P})| = 2$ and $O(e_x) = O(e_y) \neq O(e_z) = O(e_w)$

We denote $O(e_x) = O(e_y)$ by o_1, and $O(e_z) = O(e_w)$ by o_2. Since G is simple, $N(e_x) \neq N(e_y)$ and $N(e_z) \neq N(e_w)$. Hence $|N(\mathcal{P})| \geq 2$.

First, let us consider the case where $\{x, y\} \subseteq N(\mathcal{P})$, i.e., $N(e_u) = u$ for $u \in \{x, y\}$. Let $u \in \{x, y\}$ and $v \in \{z, w\}$. Note that G contains a path v-v^*-u-o_1. There exists no edge joining v and o_1 since otherwise, there exists another set $\mathcal{P}'$ of four edge-disjoint paths three of which end at o_1 with $|O(\mathcal{P}')| = 2$. Hence the P_4 has an edge uv as a chord. For applying this argument for all pairs of $u \in \{x, y\}$ and $v \in \{z, w\}$, we can observe that $G[\{v^*, x, y, z, w\}]$ contains a W_4 (See Fig. 6(a)). Also when $\{z, w\} \subseteq N(\mathcal{P})$, the existence of W_4 can be proven similarly.

Next, let us consider the case where $|N(\mathcal{P}) \cap \{x, y\}| = |N(\mathcal{P}) \cap \{z, w\}| = 1$. For example, let $N(\mathcal{P}) = \{x, z\}$, i.e., $N(e_x) = N(e_w) = x$ and $N(e_z) = N(e_y) = z$. Let $u \in \{x, z\}$ and $v \in \{y, w\}$. Observe that there exist a P_4 v-v^*-u-o_1. There exists no edge joining u and o_1 since otherwise, it contradicts to the minimality of $\mathcal{P}$. Hence the P_4 has an edge uv as a chord. For applying this argument for all pairs of $u \in \{x, z\}$ and $v \in \{y, w\}$, we can observe that $G[\{v^*, x, y, z, w\}]$ contains a W_4 (See Fig. 6(b)).

When $|O(\mathcal{P})| = 2$ and $O(e_x) \neq O(e_y) = O(e_z) = O(e_w)$

We denote $O(e_x)$ by o_1 and $O(e_y) = O(e_z) = O(e_w)$ by o_2. Since G is simple, $N(e_y)$, $N(e_z)$, and $N(e_w)$ are all different. Hence $|N(\mathcal{P})| \geq 3$.

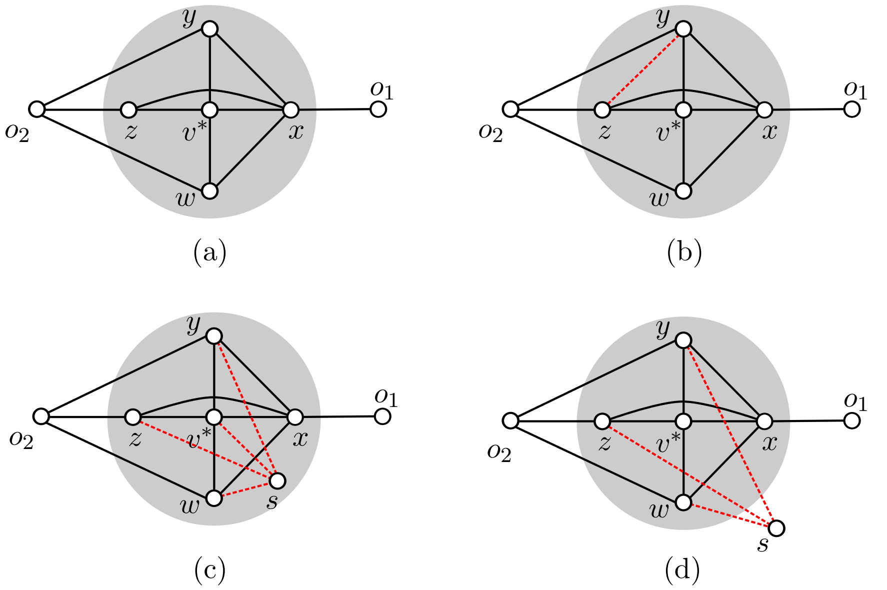


Figure 7: Examples for $|O(\mathcal{P})| = 2$, $O(e_x) \neq O(e_y) = O(e_z) = O(e_w)$ and $|N(\mathcal{P})| = 4$

First, let us consider the case where $|N(\mathcal{P})| = 4$. Then $e_x = xo_1$, $e_y = yo_2$, $e_z = zo_2$ and $e_w = wo_2$. By the choice of $\mathcal{P}$, edge xo_2 does not exist since otherwise, replacing P_x by another path v^* - x - o_2 decreases $|O(\mathcal{P})|$. For forbidding paths x - v^* - y - o_2, x - v^* - z - o_2 and x - v^* - w - o_2 from being induced subgraphs, G must contain edges xy, xz and xw (See Fig. 7(a)).

Since each vertex has at least four neighbors, y has another neighbor than v^*, x and o_2. If the neighbor is z or w, then $G[\{v^*, x, y, z, w, o_2\}]$ contains an H_1 (See Fig. 7(b)). Hence let the neighbor be another vertex, call s. If $s \in N(v^*)$, i.e., G contains an edge v^*s, then no edge joins s and o_2 since otherwise replacing P_x with another path v^* - s - o_2 decreases $|O(\mathcal{P})|$. In this case, for forbidding paths s - v^* - z - o_2 and s - v^* - w - o_2 from being induced subgraphs, there must be edges zs and ws (See Fig. 7(c)). On the other hand, if $s \in O$, then for forbidding paths z - v^* - y - s and w - v^* - y - s from being induced subgraphs, there must be edges zs and ws (See Fig. 7(d)). Therefore $G[\{v^*, x, y, z, w, o_2, s\}]$ contains an H_2 in both cases.

Next, let us consider the case where $|N(\mathcal{P})| = 3$. If $N(e_x) \in \{y, z, w\}$, then exchange e_x of P_x for $e_{N(e_x)}$ of $P_{N(e_x)}$. By this, we can assume without loss of generality that $e_x = xo_1$, $e_y = yo_2$, $e_z = zo_2$ and $e_w = xo_2$. Since G does not contain edge wo_2 by the minimality of $\mathcal{P}$, for forbidding paths w - v^* - x - o_2, w - v^* - y - o_2 and w - v^* - z - o_2 from being induced subgraphs, G must contain edges xw, yw and zw. If there is either xy or xz, then $G[\{v^*, x, y, z, w, o_2\}]$ contains an H_1 (see Fig. 8(a)). Suppose otherwise; i.e., G does not contain xy nor xz. Then, for forbidding paths y - v^* - x - o_1 and z - v^* - x - o_1 from being induced paths, there must be edges yo_1 and zo_1. In this case, $G[\{v^*, x, y, z, w, o_1, o_2\}]$ contains an H_2 (see Fig. 8(b)).

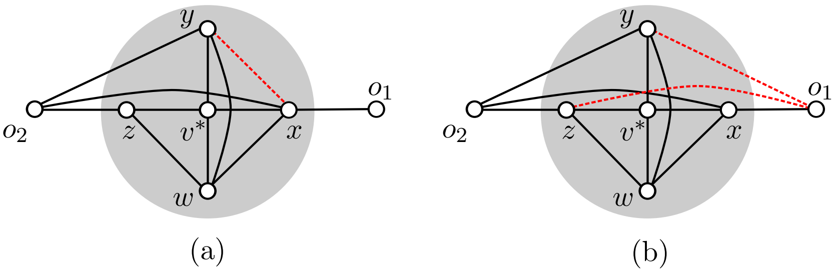


Figure 8: Examples for $|O(\mathcal{P})| = 2$, $O(e_x) \neq O(e_y) = O(e_z) = O(e_w)$ and $|N(\mathcal{P})| = 3$

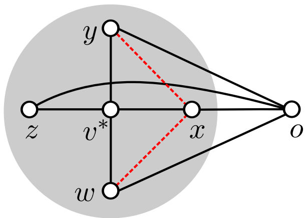


Figure 9: An example for $|O(\mathcal{P})| = 1$ and $s, t \in \{y, z, w\}$

When $|O(\mathcal{P})| = 1$

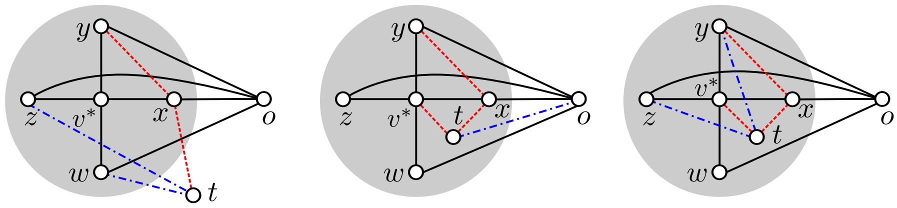
We denote the vertex in $O(\mathcal{P})$ by o. In this case, all of $N(e_x)$, $N(e_y)$, $N(e_z)$ and $N(e_w)$ are different vertices because G is simple. Thus by Claim 3, $P_x = v^*\text{-}x\text{-}o$, $P_y = v^*\text{-}y\text{-}o$, $P_z = v^*\text{-}z\text{-}o$ and $P_w = v^*\text{-}w\text{-}o$ hold. Since G is simple and 4-edge-connected, x has at least two neighbors other than v^* and o. Call two of them s and t.

First, consider the case where we can choose both s and t from $\{y, z, w\}$ (see Fig. 9). In this case, $G[\{v^*, x, s, t, o\}]$ contains a W_4.

Next, consider the case where $s \in \{y, z, w\}$ (say $s = y$) and $t \notin \{y, z, w\}$. Notice that G does not contain edges xz and xw since otherwise, this case can be reduced to the case with $s, t \in \{y, z, w\}$. If $t \in O$, then G contains edges tz and tw for forbidding paths $z\text{-}v^*\text{-}x\text{-}t$ and $w\text{-}v^*\text{-}x\text{-}t$ from being induced subgraphs. Then $G[\{v^*, x, y, z, w, t, o\}]$ contains an H_2 (see Fig. 10(a)). Hence suppose otherwise; i.e., $t \in N(v^*)$. If G contains edge to, then $G[\{v^*, x, y, t, o\}]$ contains a W_4 (see Fig. 10(b)). If G does not contain edge to, then G must contain edges ty, tz and tw for forbidding paths $t\text{-}v^*\text{-}y\text{-}o$, $t\text{-}v^*\text{-}z\text{-}o$ and $t\text{-}v^*\text{-}w\text{-}o$ from being induced subgraphs. Then $G[\{v^*, x, y, z, w, t, o\}]$ contains an H_2 (see Fig. 10(c)).

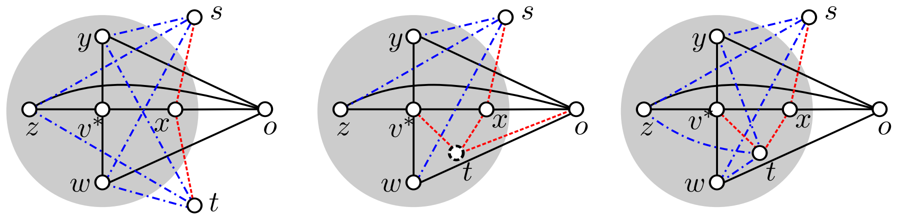
Next, consider the case where $s, t \notin \{y, z, w\}$. Notice that in this case, G does not contain edges xy, xz and xw since otherwise, this case can be reduced to the above cases. As in the previous case, we can observe the following facts:

- If $v \in \{s, t\}$ is in O, then G contains edges vy, vz and vw for forbidding paths $y\text{-}v^*\text{-}x\text{-}v$, $z\text{-}v^*\text{-}x\text{-}v$, $w\text{-}v^*\text{-}x\text{-}v$ from being induced subgraphs (type 1);

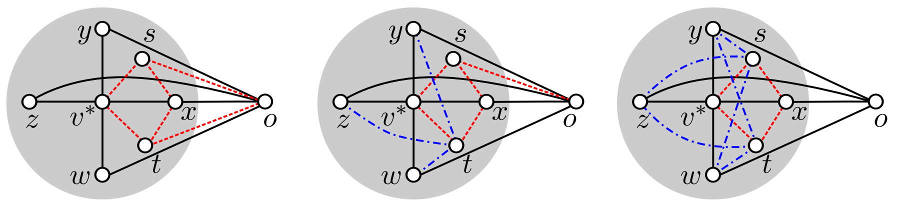


(a) (b) (c)

Figure 10: Examples for $|O(\mathcal{P})| = 1$, $s \in \{y, z, w\}$ and $t \notin \{y, z, w\}$



(a) (b) (c)



(a) (b) (c)

Figure 11: Examples for $|O(\mathcal{P})| = 1$ and $s, t \notin \{y, z, w\}$

- If $v \in \{s, t\}$ is in $N(v^*)$, then G contains either edge vo (type 2), or all of edges vy, vz and vw (type 3).

When s is type 1, we have three cases according to the type of t. These cases are illustrated in Fig. 11(a), (b) and (c). If t is type 1 (Fig. 11(a)) or 3 (Fig. 11(c)), we obtain a trivial graph after splitting off $\{v^*x, v^*y\}$ and $\{xs, ys\}$, and contracting generated C_2's repeatedly. If t is type 2 (Fig. 11(b)), $G[\{v^*, x, y, t, o\}]$ becomes a W_4 after splitting off $\{xs, ys\}$. After contracting this W_4 and repeatedly contracting generated C_2's, we obtain a trivial graph. Therefore Theorems 2 and 7 imply that $G[\{v^*, x, y, z, w, s, t, o\}]$ is $\mathbb{Z}_3$-connected regardless of the type of t.

If both s and t are type 2, then $G[\{v^*, x, s, t, o\}]$ contains a W_4 (see Fig. 11(d)). If s is type 2 and t is type 3, then $G[\{v^*, x, y, s, t, o\}]$ contains an H_1 (see Fig. 11(e)). If both s and t are type 3, then $G[\{v^*, y, w, s, t\}]$ contains a W_4 (see Fig. 11(f)). We can observe that G contains a $\mathbb{Z}_3$-connected subgraph in all cases.

4 Conclusion

We have proven the $\mathbb{Z}_3$-connectivity of 4-edge-connected HHD-free graphs. Our proof is based on the observation that every 4-edge-connected HHD-free graph contains a $\mathbb{Z}_3$-connected subgraph.

Acknowledgements

This work was partially supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan. The author thanks anonymous referees for their insightful comments.

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